Zoom Class 13/5/2020

Ques Evaluate $\lim_{n\to\infty}\left\{\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{1}{8n}\right\}$.

Soln Here the series is

$$\lim_{n\to\infty}\left\{\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{n^2\cdot 8n}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{(2n)^3}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{(n+n)^3}\right\}$$

$$\therefore t_r=\frac{n^2}{(n+r)^3}=\frac{1}{n}\left\{\frac{n^3}{(n+r)^3}\right\}$$

$$=\frac{1}{n}\left\{\frac{1}{(1+r/n)^3}\right\}$$

$\therefore$ the series becomes $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left\{\frac{1}{(1+r/n)^3}\right\}$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)=\int_0^1 f(x)\,dx$$

$$=\int_0^1\frac{1}{(1+x)^3}dx=\int_0^1(1+x)^{-3}dx=\frac{1}{-2}\left[(1+x)^{-2}\right]_0^1$$

$$=\frac{-1}{2}\left[\frac{1}{(1+x)^2}\right]_0^1=\frac{-1}{2}\left[\frac{1}{(2)^2}-\frac{1}{1^2}\right]=\frac{1}{2}-\frac{1}{8}=\frac{3}{8}$$

Ans

Ques Evaluate $\lim_{n\to\infty}\left\{\frac{n^2}{(n^2+1^2)^{3/2}}+\frac{n^2}{(n^2+2^2)^{3/2}}+\dots+\frac{n^2}{(n^2+n^2)^{3/2}}\right\}$

Soln Here $t_r = \frac{n^2}{(n^2+r^2)^{3/2}}$

$$= \frac{1}{n}\left\{\frac{n^3}{(n^2+r^2)^{3/2}}\right\} = \frac{1}{n}\left\{\frac{1}{\{1+(\frac{r}{n})^2\}^{3/2}}\right\}$$

∴ Given series is

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{1}{(1+(r/n)^2)^{3/2}}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$$

$$= \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{(1+x^2)^{3/2}}dx$$

Substituting $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$
if $x=0 \Rightarrow \theta = 0$, $x = 1 \Rightarrow \theta = \pi/4$

$$\Rightarrow \int_0^{\pi/4}\frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}} = \int_0^{\pi/4}\frac{\cancel{\sec^2\theta}}{\sec^3\theta}d\theta$$

$$= \int_0^{\pi/4}\cos\theta\, d\theta = [\sin\theta]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$ **Ans**

Ques Evaluate $\lim_{n\to\infty}\left\{\frac{1}{\sqrt{2n-1}}+\frac{1}{\sqrt{4n-4}}+\frac{1}{\sqrt{6n-9}}+\cdots+\frac{1}{n}\right\}$

Soln
Given series can be written as

$$\lim_{n\to\infty}\left\{\frac{1}{\sqrt{2n-1}}+\frac{1}{\sqrt{4n-4}}+\frac{1}{\sqrt{6n-9}}+\cdots+\frac{1}{\sqrt{n^2}}\right\}$$

Also $\frac{1}{\sqrt{n^2}}=\frac{1}{\sqrt{2n^2-n^2}}=\frac{1}{\sqrt{n(2n)-n^2}}$

$\therefore$ the series becomes

$$\lim_{n\to\infty}\left\{\frac{1}{\sqrt{1(2n)-1^2}}+\frac{1}{\sqrt{2\cdot(2n)-2^2}}+\frac{1}{\sqrt{3(2n)-3^2}}+\cdots+\frac{1}{\sqrt{n(2n)-n^2}}\right\}$$

$$\therefore t_r=\frac{1}{\sqrt{r(2n)-r^2}} \Rightarrow \frac{1}{n}\left\{\frac{n}{\sqrt{r(2n)-r^2}}\right\}$$

$$=\frac{1}{n}\left\{\frac{1}{\sqrt{2\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2}}\right\}$$

$\therefore$ Series is $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left\{\frac{1}{\sqrt{2\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2}}\right\}$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^1 \frac{dx}{\sqrt{1-(1-2x+x^2)}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1^2-(1-x)^2}}$$

Substituting $(1-x) = y \Rightarrow -dx = dy$

If $x \to 0 \Rightarrow y \to 1$, If $x \to 1 \Rightarrow y \to 0$

$$\therefore I = \int_1^0 \frac{-dy}{\sqrt{1^2-y^2}} = \int_0^1 \frac{dy}{\sqrt{1^2-y^2}}$$

$$= \left[\sin^{-1} y\right]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2} \quad \underline{\text{Ans}}$$

Ques Evaluate $\lim_{n\to\infty} \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{4}{n^2}\right)\left(1+\frac{9}{n^2}\right)\cdots(2)\right\}^{1/n}$

Soln Given series can be written as

$$\lim_{n\to\infty} \left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n}\right)\cdots(1+1)\right\}^{1/n}$$

$$\Rightarrow \lim_{n\to\infty} \left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n} = Z \text{ (let)}$$

$$\Rightarrow \log z = \lim_{n\to\infty} \left[\log\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$\therefore t_r = \frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right) \qquad \hookrightarrow \begin{bmatrix}\log m^n = n\log m\\ \log mn = \log m + \log n\end{bmatrix}$$

$$\therefore \log z = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

$$\Rightarrow \log z = \int_0^1 \log(1+x^2)\,dx$$

Integrating by parts we get

$$\int \log(1+x^2)\,dx = x\cdot\log(1+x^2) - \int \frac{1}{1+x^2}\,2x\cdot x\,dx$$

$$= x\log(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$= x\log(1+x^2) - 2\int \frac{(1+x^2)-1}{(1+x^2)}\,dx$$

$$= x\log(1+x^2) - 2\int dx + 2\int \frac{dx}{1+x^2}$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x.$$

Putting limits we get

$$= \left[x\log(1+x^2)\right]_0^1 - 2\left[x\right]_0^1 + 2\left[\tan^{-1}x\right]_0^1$$

$$\Rightarrow \log z = [\log 2 - 0] - 2 + 0 + 2\tan^{-1} x - 2\tan^{-1} 0$$

$$= \log 2 - 2 + 2\cdot\frac{\pi}{4} - 0$$

$$= \log 2 - 2 + \frac{\pi}{2}$$

$$\Rightarrow z = e^{(\log 2 - 2 + \pi/2)}$$

$$= e^{\log 2}\cdot e^{-2}\cdot e^{\pi/2} = 2\cdot\frac{1}{e^2}\cdot e^{\pi/2}$$

$$\Rightarrow z = 2e^{\pi/2 - 2}$$

Ans